# Integrals

## Multiple Choice Questions

**Q: 1** **Look at the integral given below.**

$$\int_{\frac{a}{4}}^{\frac{b}{4}} f(4x)dx$$

**If *f* ( *x* ) is continuous for all real values of *x*, then which of these is equal to the above integral?**

1. $4\int_a^b f(x)dx$
2. $\frac{1}{4}\int_a^b f(x)dx$
3. $\frac{1}{4}\int_{4a}^{4b} f(x)dx$
4. $4\int_{4a}^{4b} f(x)dx$

**Q: 2** **What is the value of the following integral?**

$$\int_{-2}^{2} (2 - |x|)\, dx$$

1. 0
2. 4
3. 8
4. 12

**Q: 3** **Varath says the following:**

$$\int_{-3}^{3} \left(\sqrt{x^2 - 4}\right) dx = F(3) - F(-3),$$

where $F(x)$ is the antiderivative of $\left(\sqrt{x^2 - 4}\right)$.

**Which of the following can be said about Varath's statement?**

1. It is true, as the function is continuous in [-3,3].
2. It is true, as per the fundamental theorem of calculus.
3. It is false, as the integral is not defined over the interval.
4. It is false, as the antiderivative of any function within a square root does not exist in R.

## Free Response Questions

**Q: 4** **Integrate the following function with respect to *x*.** **[1]**

$$e^{6-\ln x}$$

**Show your steps.**

**Q: 5** **Given:** **[1]**

$$\int \frac{dx}{f(x)} = \frac{1}{3}\tan^{-1}\left(\frac{x-4}{3}\right) + C$$

**where C is an arbitrary constant.**

**Find *f* ( *x* ). Show your work.**

**Q: 6** Check whether the given statement is true or false. **[1]**

For any function $f(x)$ that satisfies the condition $f(-3) = -f(3)$,
$\int_{-3}^{3} f(x) = 0$

Justify your answer.

**Q: 7** **Ankit's partial solution for a question on integration is given below.** **[2]**

Question: Solve $\int \cos^3 x \sin^2 x \, dx$.

Ankit's solution:
Step 1: Let I = $\int \cos^3 x \sin^2 x \, dx$
Step 2: I = $\int \cos^3 x(1 - \cos^2 x) \, dx$
Step 3: I = $\int (\cos^3 x - \cos^5 x) \, dx$
Step 4: Let $\cos x = t$
Step 5: I = $\int (t^3 - t^5) \, dt$

**Is Ankit correct? If yes, complete the integration. If no, in which step is the error present? Explain your reasoning.**

**Q: 8** **If *h* '( *x* ) = *g* ( *x* ) and *g* is a continuous function for all real values of *x*, then prove that:** **[2]**

$$\int_{-1}^{1} g(6x)dx = \frac{1}{6}h(6) - \frac{1}{6}h(-6)$$

**Q: 9** **Solve:** **[3]**

$$\int \frac{1}{m^2} \cos^2\left(\frac{1}{m} - 1\right) dm$$

**Show your steps.**

**Q: 10** **Evaluate the integral:** **[3]**

$$I = \int_0^{\frac{\sqrt{3}}{2}} \left[\frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx\right]$$

**Show your work.**

**Q: 11** **Solve the following integration and write your answer in its most simplified form.** **[5]**

$$\int \frac{x^2}{e^{2x}} dx$$

**Show your steps.**

**Q: 12** **Integrate the given function. Show your steps.** **[5]**

$$\int \cot^{-1}\left(\frac{5}{x}\right) dx$$

**Q: 13** Integrate $\frac{2e^x}{(e^x-1)(2e^x+3)}$. **[5]**

Show your work.

**The table below gives the correct answer for each multiple-choice question in this test.**

| Q.No | Correct Answers |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 3 |

| Q.No | What to look for | Marks |
|---|---|---|
| 4 | **Rewrites the given expression as:** $\frac{e^6}{x}$ | **0.5** |
| | **Integrates the above expression with respect to *x* as:** $e^6 \ln\|x\| + C$ **where, C is the constant of integration.** | **0.5** |
| 5 | **Finds *f* ( *x* ) as:** $9 + (x - 4)^2$ **by differentiating the RHS of the given equation using the differentiation of** $\tan^{-1} x$ **as follows:** $\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}\left(\frac{x-4}{3}\right)\right) = \frac{1}{9+(x-4)^2}$ | **1** |
| 6 | **Writes false.** | **0.5** |
| | **Reasons that this would only be true if** $f(-x) = -f(x)$ **for every value of *x.*** | **0.5** |
| 7 | **Writes that Ankit is incorrect.** | **0.5** |
| | **Writes that Ankit made an error in step 5.** | **0.5** |
| | **Gives a reason. For example, on substituting cos *x* as *t*, one also has to substitute *dx* in terms of *dt,* which Ankit has not done. Instead, he has simply replaced *dx* with *dt*.** | **1** |
| 8 | **Substitutes** $u = 6x$ **and finds *du* as 6 *dx*.** | **0.5** |
| | **Finds the limit as:** **When** $x = -1$, **then** $u = -6$. **When** $x = 1$, **then** $u = 6$. | **0.5** |

| Q.No | What to look for | Marks |
|---|---|---|
| | **Rewrites the integral and proves the given statement as:**<br><br>$\int_{-1}^{1} g(6x)dx$<br>$= \frac{1}{6}\int_{-6}^{6} g(u)du$<br>$= \left[\frac{1}{6}h(u)\right]_{-6}^{6}$<br>$= \frac{1}{6}h(6) - \frac{1}{6}h(-6)$ | 1 |
| 9 | **Substitutes ( $\frac{1}{m}$ - 1) as *u* to get:**<br><br>$du = -\frac{1}{m^2}dm$ | 0.5 |
| | **Rewrites the given integral as:**<br><br>$-\int \cos^2 u \, du$ | 0.5 |
| | **Substitutes:**<br><br>$\cos^2 u = \frac{(1+\cos 2u)}{2}$<br><br>**in the above integral and rewrites it as follows:**<br><br>$-\frac{1}{2}\int (1 + \cos 2u)\, du$ | 0.5 |
| | **Integrates the above integral to get the following expression where C is the arbitrary constant:**<br><br>$-\frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right) + C$ | 1 |

| Q.No | What to look for | Marks |
|---|---|---|
| | **Substitutes $u$ as ( $\frac{1}{m}$ - 1) in the above expression to get the following expression as the solution:** <br> $-\frac{1}{2}\left[\frac{1}{m}-1+\frac{1}{2}\sin 2\left(\frac{1}{m}-1\right)\right]+C$ | 0.5 |
| 10 | **Takes $u = \sin^{-1} x$** <br> **Finds du as:** <br> $du = \frac{dx}{\sqrt{1-x^2}}$ | 0.5 |
| | **Finds the change in limit when $x = 0$ and $x = \frac{\sqrt{3}}{2}$ to $u = 0$ and $u = \frac{\pi}{3}$ respectively.** | 0.5 |
| | **Rewrites the given integral using the above substitution and integrates the same as:** <br> $I = \int_0^{\frac{\pi}{3}} \frac{u}{\cos^2 u} du$ <br> $\Rightarrow I = \int_0^{\frac{\pi}{3}} u \sec^2 u \, du$ <br> $\Rightarrow I = u \tan u \,\Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan u \, du$ <br> $\Rightarrow I = u \tan u \,\Big|_0^{\frac{\pi}{3}} + \log \cos u \,\Big|_0^{\frac{\pi}{3}}$ | 1.5 |
| | **Applies the limit to find the value of the given definite integral as:** <br> $\frac{\pi}{\sqrt{3}} - \log 2$ | 0.5 |
| 11 | Let $I = \int \frac{x^2}{e^{2x}} dx$ <br> **Evaluates integral using formula for integral by parts to get the following:** <br> $I = x^2\left(\int e^{-2x} dx\right) - \int \left[2x \int e^{-2x} dx\right] dx$ | 1 |

| Q.No | What to look for | Marks |
|---|---|---|
| | **Solves the integration of $e^{-2x}$ by substituting $u$ as $(-2x)$.**<br><br>**Gets $dx$ as $\frac{-1}{2}du$.**<br><br>**The integration may look as follows:**<br><br>$-\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c$<br>$\Rightarrow \int e^{-2x}dx = -\frac{1}{2}e^{-2x} + c$<br><br>**where c is a constant.** | 1.5 |
| | **Substitutes the value of the above integral in the equation from step 2 to get the following:**<br><br>$I = -\frac{x^2}{2e^{2x}} + \int xe^{-2x}dx$ | 0.5 |
| | **Applies integration by parts to solve the integration of $xe^{-2x}$ in a similar way as in step 3.**<br><br>**The integration may look as follows:**<br><br>Let $h(x) = x$ and $g(x) = e^{-2x}$<br><br>$\Rightarrow \int xe^{-2x}dx = \frac{x}{-2e^{2x}} - \int\left[\int e^{-2x}dx\right]dx$<br>$\Rightarrow \int xe^{-2x}dx = -\frac{x}{2e^{2x}} + \frac{1}{2}\int e^{-2x}dx$<br>$\Rightarrow \int xe^{-2x}dx = -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}} + c_1$<br><br>where $c_1$ is a constant | 1.5 |
| | **Writes answer as follows:**<br><br>$I = -\frac{2x^2+2x+1}{4e^{2x}} + C$<br><br>**where C is a constant.** | 0.5 |

| Q.No | What to look for | Marks |
|---|---|---|
| 12 | **Uses integration by parts to rewrite the given integral as:**<br><br>$\cot^{-1}\left(\frac{5}{x}\right)\int dx - \int\left[\frac{d}{dx}\left(\cot^{-1}\left(\frac{5}{x}\right)\right)\int dx\right]dx$ | 1 |
| | **Simplifies the differentiation of $\cot^{-1}\left(\frac{5}{x}\right)$ in the above expression as:**<br><br>$\frac{d}{dx}\cot^{-1}\left(\frac{5}{x}\right)$<br>$= -\frac{1}{1+\left(\frac{5}{x}\right)^2}\frac{d}{dx}\left(\frac{5}{x}\right)$<br>$= \frac{\frac{5}{x^2}}{1+\left(\frac{5}{x}\right)^2}$<br>$= \frac{5}{x^2+25}$ | 1.5 |
| | **Substitutes the above expression in step 1 and integrates the integral in step 1 to get the following expression:**<br><br>$x\cot^{-1}\left(\frac{5}{x}\right) - 5\int\frac{x}{x^2+25}dx$ | 1 |
| | **Completes integrating the above expression to get:**<br><br>$x\cot^{-1}\left(\frac{5}{x}\right) - \frac{5}{2}\log\left\|x^2+25\right\| + C$<br><br>**where C is an arbitrary constant.**<br><br>**(Award full marks even if modulus is not used in log function as ( $x^2$ + 25) is always positive.)** | 1.5 |
| 13 | Considers $e^x = t$ and finds $dx = \frac{dt}{t}$. | 1 |

| Q.No | What to look for | Marks |
|---|---|---|
| | Writes the function to be integrated as $\int \frac{2}{(t-1)(2t+3)}dt$. | 1 |
| | **Uses partial fractions to rewrite the function as follows:**<br>$\frac{2}{5}\int\left(\frac{1}{t-1}-\frac{2}{2t+3}\right)dt$ | 1.5 |
| | **Integrates the function to obtain the following:**<br>$\frac{2}{5}[\ln\|t-1\|-\ln\|2t+3\|]+C$<br>**where C is the constant of integration.** | 1 |
| | Replaces $t$ with $e^x$ to obtain $\frac{2}{5}\ln\left\|\frac{e^x-1}{2e^x+3}\right\|+C$. | 0.5 |